The Estimate Of The Definite Integral By Monte Carlo Method

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Abstract: In this paper, the Monte Carlo method, namely the method of random value, through the MATLAB program, give the estimated value of definite integral and double integral. Furthermore, the corresponding error analysis is given by comparing with the exact value.

1. Introduction

Monte Carlo method was first proposed by S.M. Ulam and J. von Neumann in the 1940s.Its core idea is to adopt statistical simulation method. With the development of electronic computer technology, Monte Carlo method has been widely applied in financial engineering, computational physics, artificial intelligence and other fields [1-4]. The thought of calculus is the core content of higher mathematics. The integral is mainly divided into four steps: arbitrary segmentation interval, arbitrary values within the cell quadrature, sum, take the limit. Due to the arbitrariness of the segmentation interval and the arbitrariness of the value of the integral definition, the estimation value of the definite integral can be obtained by using the methods of random segmentation and random value [5], that is, Monte Carlo method can be used to simulate the calculation of the definite integral, which can help us understand the definite integral more deeply in essence.

2. The Simulation Of The Definite Integral By Monte Carlo Method

First, the concept of definite integral is given .

Definition 1 [6] Let a bounded function $f(x)$ is defined in $[a,b]$, it can be made into a dividing method P: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ by $\{x_i\}_{i=0}^{n}$, to mark the length between $[x_{i-1}, x_i]$ cells as $\Delta x_i = x_i - x_{i-1}$.

And remember $\lambda = \max\limits_{1 \le i \le n}(\Delta x_i)$, if $\forall \xi_i \in [x_{i-1}, x_i]$ and the limit $\lim\limits_{\lambda \to 0} \sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$ exists, and the value of the limit is independent of both the division method P and the taking method of ξ_i, then it is said to $f(x)$ be Riemann, integrable in upper $[a,b]$ and the sum $s_n = \sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$ is called Riemann sum, and the limit value I is called the definite integral in upper $[a,b]$, denoted as $I = \int_a^b f(x)dx$.

Steps of monte Carlo simulation of $I = \int_a^b f(x)dx$.

Step 1: Given (small enough) λ.

Step 2: Mark a random value as x_1 in upper $(a, a+\lambda]$, and mark a random value asi x_2 n upper $(x_1, x_1+\lambda]$.In turn, a random value is denoted as x_{i+1} in upper $(x_i, x_i+\lambda]$, until $x_{i+1} \ge b$, at this time, let $x_{i+1} = b$,then P: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

Step 3: Random value ξ_i between each cell $[x_{i-1}, x_i]$.

Step 4: Sum $s_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$, which can be used as the simulation value $I = \int_a^b f(x)dx$.

Example 1: Evaluate the definite integral $\int_0^1 e^{x^2} dx$.

Matlab program is as follows:

```
syms a b c lamd n;
a = 0; b = 1; lamd = 0.0001; % Given tne bounds of the integeal and the value of λ.
P = a (1); c= a;
i = 1;
While c < b
P (i + 1) = unifrnd (P(i), P (i) + lamd);
i = i+ 1; c = P (i - 1);
end
P;    %Generate random interval.
n = length (P);
for i = 1: (n - 1)
        Q (i) = unifrnd (P (i), P (i + 1)); %Take random values.
end
Q;
sum = 0;
for i = 1: (n - 1)
        sum = sum + exp (Q (i) ^ 2) * (P (i + 1) - P (i));    % Sum.
end
sum
```

Run result: sum= 1.4627

The exact value of $\int_0^1 e^{x^2} dx$ is as follows:

The vpa (int (' exp (x ^ 2), 0, 1))

ans = 1.46265

Here are the different estimates of $\int_0^1 e^{x^2} dx$ for different values λ.

Table 1. The estimates of $\int_0^1 e^{x^2} dx$

lamd	0.1	0.01	0.001	0.0001
Estimate of the integral	1.7993	1.4838	1.4637	1.4627
error	0.33665	0.02115	0.00105	0.00005
The relative error	0.2302	0.0145	0.0007	0.00003

We can extend the above method to estimate the double integral. The definition of double integral is as follows:

Definition 2[7-9] Set $f(x, y)$ as a bounded function defined on a closed region D.

(1) Segmentation: Divide D by any two groups of curves into n small pieces $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n$.

(2) Make a sum: Take any $(\xi_i, \eta_i) \in \Delta\sigma_i$ and make a sum $\sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$.

(3)Take the limit: $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$ (where λ represents the maximum diameter of each small piece).

If this limit exists and does not depend on the division method of the region D, nor does it depend on the taking method of the point (ξ_i, η_i), then it is said that the function $f(x, y)$ is integrable on the region D, and the limit value is said to be the double integral of $f(x, y)$ over D, denoted as $\iint\limits_D f(x,y)d\sigma$.

Example 2 Estimates the double integral of $\iint\limits_D \frac{\sin(x+y)}{x+y+1}d\sigma$, where $D: 0 \le x \le 1, 0 \le y \le 1$

Matlab program is as follows:

```
syms a b c d e f   lamd   n   m;
a= 0; b = 1;d = 0; e = 1; lamd = 0.001; % Given tne bounds of D  and the value of λ.
P(1) = a; c= a;
i= 1;
While c < b
P (i + 1) = unifrnd (P (i), P (i) + lamd); % Random division of D  by vertical lines
i = i + 1; c = P (i - 1);
  end
P;
n = length (P);
Q (1) = d;f= d;
j= 1;
While f < e
Q (j + 1) = unifrnd (Q (j), Q (j) + lamd); %Random division of D through horizontal   lines.

j = j + 1;f = Q (j - 1);
end
Q;
m = length (Q);
for i= 1: (n - 1)
          for j = 1: (m - 1)
      X = unifrnd (P (i), P (i + 1));
      Y = unifrnd (Q (j), Q (j + 1)); %Take random values.
Z (i, j) = (sin(x + y))/(x + y + 1);
  end
end
sum = 0;
for i = 1: (n - 1)
            for j = 1: (m - 1)
sum = sum + Z (i, j) * ((P (i + 1) - P (i)) * (Q (j + 1) - Q (j)));   % Sum.
end
  end
sum
```

Run result: sum =0.3800

Matlab program [10] to evaluate the exact value:

```
syms x y
x0 = 0; x1 = 1; y0 = 0;y1 = 1;
fun = (sin (x + y))/(x + y + 1);
a = int (int (fun, y, y0, y1), x, x0, x1);
b = vpa (a)
```

The exact value B =0.379073539

The following is the estimated value of $\iint\limits_D \frac{\sin(x+y)}{x+y+1}d\sigma$ and error analysis of the double integral when λ are0.1, 0.01 and 0.001 respectively.

Table 2. The estimates of $\iint_D \frac{\sin(x+y)}{x+y+1} d\sigma$

λ	0.1	0.01	0.001
Estimate of the integral	0.4459	0.3825	0.3800
error	0.06682646	0. 00342646	0.00092646
The relative error	0. 1762889	0.009039039	0.00244401

Conclusion

Monte Carlo method provides a new method for definite integrals, which is simple and easy, and can help us to understand the nature of definite integrals more deeply.

When λ is very small, the Matlab program calculation is relatively large. In addition, in the absence of accurate values, the estimation of errors still needs to be further studied.

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